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# A New Interval-Based Method to Characterize Estimability

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SUMMARY

Estimability is a property which states on the accuracy of the parameter estimation in the case of experimental data. This paper defines a new method based on interval analysis and set inversion to characterize estimability in the case of a bounded additive noise. To illustrate this new method, the Time Difference of Arrival (TDOA) passive location estimability is evaluated: to our knowledge, it is the first time that the parameter estimation error of these nonlinear equations is given. 

KEY WORDS: Estimability, Identifiability, Interval Analysis, nonlinear Models, Experimental Design.

## 1. Introduction

Estimability is a property which states on the accuracy of the parameter estimation in the case of experimental data [1, 2]. Indeed, a parameter can be identifiable [3, 4] but poorly estimable

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for a given experiment. This paper defines a new method based on interval analysis and set inversion to characterize estimability in the case of a bounded additive noise.

A bounded-error estimation problem can be written under the form [5]:

$$\mathbf{y} = \mathbf{f}(\mathbf{p}) + \mathbf{e}, \tag{1}$$

where $\mathbf{e} \in \mathbb{E}$ stands for an additive noise vector, $\mathbb{E}$ stands for the additive noise set and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a nonlinear function. Our interval-based estimability approach focuses on $\hat{\mathbf{p}}$ vectors which are estimable from $\mathbf{y}$, i.e. that can lead to the same measurement vector $\mathbf{y}$. We are looking for $\mathbb{P}$ set such as:

$$\mathbb{P} = \{\hat{\mathbf{p}} \in \mathbb{R}^n \mid \exists(\mathbf{e}_1, \mathbf{e}_2) \in \mathbb{E}^2, f(\hat{\mathbf{p}}) + \mathbf{e}_1 = \mathbf{f}(\mathbf{p}) + \mathbf{e}_2\} \tag{2}$$

Define the uncertainty set $\mathbb{U} = \{\mathbf{e}_2 - \mathbf{e}_1 | \mathbf{e}_1 \in \mathbb{E}, \mathbf{e}_2 \in \mathbb{E}\}$ and $\mathbb{Y} = \mathbf{f}(\mathbf{p}) + \mathbb{U}$. Then, $\mathbb{P}$ can be written as a set inversion [6]:

$$\mathbb{P} = f^{-1}(\mathbb{Y}). \tag{3}$$

Figure 1 illustrates our estimability approach.

In next section, we define the estimability function $\xi_f$ which characterizes the size of $\mathbb{P}$. Third section shows how interval analysis and set inversion may be used to evaluate of $\xi_f$. Finally, last section illustrates $\xi_f$ relevance by evaluating the estimability of a nonlinear passive location function.

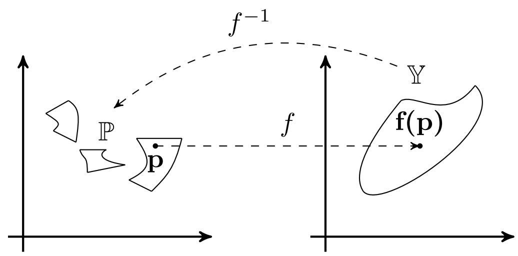



Figure 1. Illustration of our estimability approach: $\mathbb{P}$ constitutes the reciprocal image of $\mathbb{Y} = \mathbf{f(p)} + \mathbb{U}$. Our estimability function $\xi_f$ evaluates the size of $\mathbb{P}$.

## 2. Estimability Function $\xi_f$

### *2.1. Preliminary definition*

To define $\xi_f$, we need a general size function $w$ such as:

$$\begin{array}{rccc} w: & \mathcal{C}(\mathbb{R}^n) & \rightarrow & \mathbb{R}^+ \\ & \mathbb{A} & \rightarrow & w(\mathbb{A}) \end{array} \tag{4}$$

where $\mathcal{C}(\mathbb{R}^n)$ stands for compact sets of $\mathbb{R}^n$. The general size function satisfies two conditions: $w(\mathbb{A})$ always belongs to $\mathbb{R}^+$ and $w$ is monotonic, i.e. $\mathbb{A} \subseteq \mathbb{B} \Rightarrow w(\mathbb{A}) \leq w(\mathbb{B})$. Classically, $w$ is chosen as the largest dimension of the smallest box containing $\mathbb{A}$. Nevertheless, depending on the context and the dimension $n$, $w$ may account for area, volume or diameter of a compact set [7, 8].

### *2.2. Estimability Function $\xi_f$ Definition*

In the following, $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ stands for a nonlinear function. Then we can define estimability function $\xi_f$ as follow:

$$\begin{array}{rccl} \xi_f : & \mathbb{R}^n & \rightarrow & \mathbb{R}^+ \\ & \mathbf{p} & \rightarrow & w(f^{-1}(\mathbf{f(p)} + \mathbb{U})) \end{array} \tag{5}$$

where $\mathbb{U} = \{\mathbf{e}_2 - \mathbf{e}_1 | \mathbf{e}_1 \in \mathbb{E}, \mathbf{e}_2 \in \mathbb{E}\}$ is the uncertainty set and $\mathbb{E}$ stands for the additive noise set. $\xi_f(\mathbf{p})$ value is the size of the inverted set of $\mathbb{Y} = \mathbf{f(p)} + \mathbb{U}$.

### *2.3. Illustration of Estimability Function*

To illustrate $\xi_f$ concept, let us choose the following one-dimension nonlinear function $f$:

$$\begin{array}{rccl} f : & [0, 6] & \rightarrow & \mathbb{R} \\ & x & \rightarrow & \sqrt{x}\, sin(x) + x. \end{array} \tag{6}$$

This $f$ function is sketched in Fig. 2 and $\xi_f(1)$ evaluation is detailed. We suppose that the additive noise set is $[-\varepsilon/2, \varepsilon/2]$. Then, interval analysis allows us to write :

$$\mathbb{U} = [-\varepsilon/2, \varepsilon/2] - [-\varepsilon/2, \varepsilon/2] = [-\varepsilon, \varepsilon]. \tag{7}$$

In this example, we choose $\varepsilon = 0.7$. Therefore $f^{-1}(f(1) + [-\varepsilon, \varepsilon])$ results in two intervals $\mathbb{A}_1$ and $\mathbb{A}_2$. Let us denote by $a_{i_-}$ and $a_{i_+}$ the $\mathbb{A}_i$ lower and upper bound. $w$ result is the sum of the diameters of these two intervals[†]. That is why:

$$\xi_f(1) = (a_{1_+} - a_{1_-}) + (a_{2_+} - a_{2_-}).$$

$\xi_f(1)$ is found to be about 1.55. It characterizes parameter estimation error due to additive noise and nonlinearity of $f$ near $x = 1$.

The lesser $\xi_f(x)$, the better the accuracy of the parameter estimation. On the contrary, $\xi_f(x) \gg 1$ characterizes the impossibility to properly estimate parameters: it is due to noise, low growing rate or non-injectivity of $f$ [9] .

[†]This definition of $w$ is consistent with the section 2.1 of this paper.

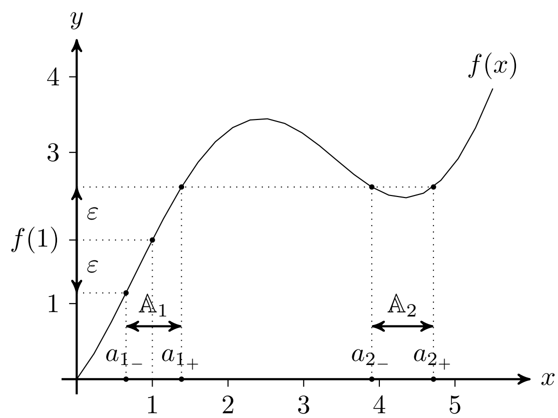



Figure 2. One dimension $f(x) = \sqrt{x}\,sin(x) + x$ function and $\xi_f$ concept: $\varepsilon/2 = 0.35$, $x = 1$. In this example, $f^{-1}(f(1) + [-\varepsilon, \varepsilon])$ results in two intervals $\mathbb{A}_1$ and $\mathbb{A}_2.\xi_f(1) = (a_{1_+} - a_{1_-}) + (a_{2_+} - a_{2_-})$.

## 3. Estimability Evaluation

### *3.1. Methodology*

To evaluate $\xi_f$, four stages are required: firstly, $\mathbb{U}$ must be deduced from $\mathbb{E}$. Secondly, $\mathbf{f(p)}+\mathbb{U}$ of (5) is evaluated. Then, $f^{-1}(\mathbb{Y})$ is characterized by using set inversion [10]. Finally, $w$ computes the sum of the sizes of the resulting intervals.

Powerful set methods exist to address set inversion problems [6]. In this paper, we are using Quimper, a high-level language for QUick Interval Modeling and Programming in a bounded-ERror context[‡]. Quimper uses interval analysis and constraint propagation [11] to solve equations. It guarantees that the computed intervals enclose all solutions for given initial intervals. In addition, it provides built-in contractors which speed up computation. Details about Quimper and contractor programming can be found in [12]. But let us now illustrate

[‡]See Ibex/Quimper site at `http://ibex-lib.org/`

the estimability function $\xi_f$ on a one dimension example.

### *3.2. 1-D Estimability Evaluation*

$\xi_f$ and $f$ of (6) are drawn for $x \in [0, 20]$ in Fig. 3. Each point of $\xi_f(x)$ has been evaluated using contractor set inversion. Quimper script for each point is similar to listing 1.

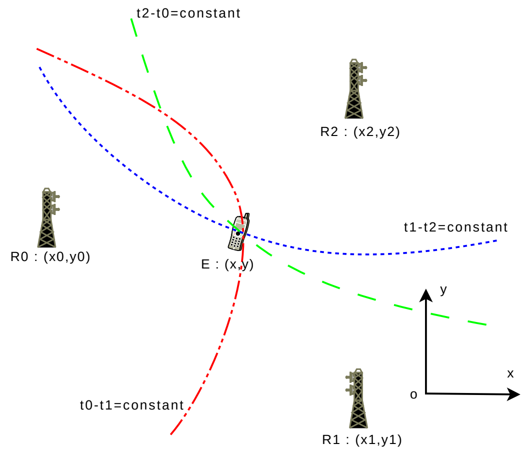
Listing 1. Example of Quimper script for set inversion

```
Constants
fxme=1.33470616479;
fxpe=2.73470616479;
Variables
x in [0,20];
contractor finv
x+sqrt(x)*sin(x) in [fxme,fxpe]
end
contractor isThick
  maxdiamGT(0.01)
end
```

In this listing 1, true value of $x$ is 1.1, $\mathbb{E} = [-\varepsilon/2, \varepsilon/2] = [-0.35, 0.35]$ and $[fxme, fxpe]$ stands for $[f(x) - \varepsilon, f(x) + \varepsilon]$. The contractor $finv$ eliminates all the intervals which do not satisfy (6). $isThick$ is a special contractor to fix the bisection limits. It collects all the intervals whose maximum size is 0.01. Therefore the intervals that are not solutions and the intervals that are indiscernible are included by contractor $isThick$[§]. Computation takes about 0.002s per point[¶].

[§]See Quimper manual for examples.
[¶]on an Intel Core 2 Duo CPU at 2.00GHz




$\xi_f$ is not monotonic over $[0, 20]$. Structural identifiability [9] tells us that it is due to variation of the cardinality of $f^{-1}(\mathbb{Y})$. $\xi_f$ can take high values because of non-injectivity. On the contrary, if the injective part of $f$ is considered and if the growing rate of $f$ is high, then $\xi_f$ tends to 0.

## 4. Application to Passive Location

### *4.1. TDOA Hyperbolic Equations*

Let $(x, y)$ be the unknown location of the emitter, and $(x_i, y_i)$ the location of the receivers. Distance from emitter to receiver $i$ is:

$$D_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{8}$$

Let $t_{ij}$ be the measured[‖] Time Difference Of Arrival (TDOA) of the signal between receiver

[‖]See [13] and [14] for correlation techniques used to measure TDOA.

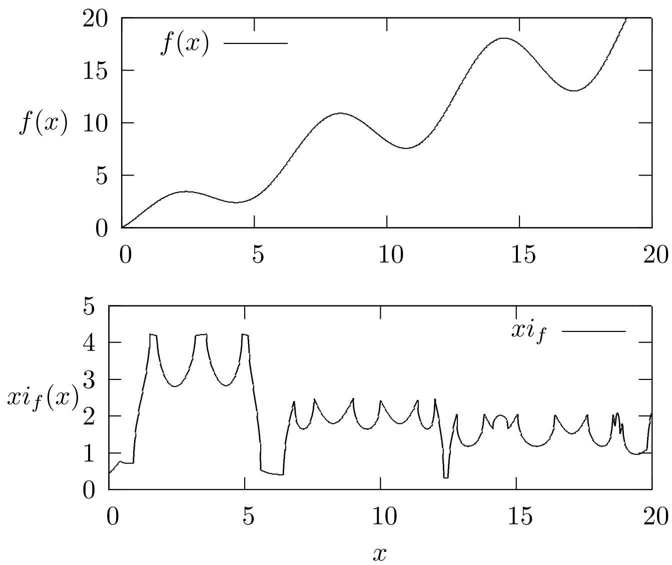



Figure 3. One dimension $f(x) = \sqrt{x} sin(x) + x$ function and $\xi_f$ over x-range $[0, 20]$ for $\varepsilon = 0.7$.

$i$ and $j$. As $D_i - D_j = ct_{ij}$, hyperbolic TDOA equations are:

$$\sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_j)^2 + (y - y_j)^2} = ct_{ij} \quad (9)$$

where $c$ is the speed of the signal and $(i, j) \in \{(0, 1), (1, 2), (2, 0)\}$.

Solving these nonlinear equations for $(x, y)$ is not a trivial problem [15, 16, 17], especially when time measurements are noisy. However, we have shown in [18] that our approach based on interval analysis, constraint propagation and contractor programming allows us to avoid any approximations and naturally results in bounded-error estimation.

### 4.2. TDOA Estimability

Consider the following function:

$$f: \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ (x, y) & \rightarrow & (t_{01}, t_{12}, t_{20}). \end{array} \tag{10}$$

where $t_{ij}$ is defined by (9). The estimability of this function allows us to refine our TDOA approach: for a given time additive noise and a special receivers configuration, we can now easily build a map which states on the TDOA passive location error.

In this example, receivers are located at R0 (-1000, 0) m, R1 (0,1000) m and R2 (1000,0) m. We choose to define $w$ as area operator. Therefore, $\xi_f$ unit is $km^2$. In this simulation, $\mathbb{E} = [-\varepsilon/2, \varepsilon/2] \times [-\varepsilon/2, \varepsilon/2] \times [-\varepsilon/2, \varepsilon/2]$ and $\varepsilon/2 = 15$ns. This time error** corresponds to an analog to digital converter with a good precision and a basic signal correlation.

A 100x100 $xy$ grid has been defined over x-range $[-5000, 5000]$ and y-range $[-5000, 5000]$. For each point $(x, y)$, a Quimper file similar to listing 2 is computed. The area corresponding to $\xi_f(x, y)$ is extracted from Quimper results. Figure 4 shows $\xi_f$ computation. Each point takes about 0.02 s to compute.

Listing 2. Example of Quimper script for TDOA set inversion

```
Constants
x0=0.0;
y0=-1000.0;
x1=0.0;
y1=1000.0;
x2=1000.0;
```

**Different $\varepsilon/2$ could have been chosen for each $t_{ij}$. They also could have been chosen randomly.

```
y2=0.0;
ct01 in [-592.12,-577.12];
ct12 in [-588.53,-573.53];
ct20 in [1158.15,1173.19];
Variables
x in [-5000,5000];
y in [-5000,5000];

constraint-list dtoaeq
sqrt((x-x0)^2+(y-y0)^2)-sqrt((x-x1)^2+(y-y1)^2) in ct01;
sqrt((x-x1)^2+(y-y1)^2)-sqrt((x-x2)^2+(y-y2)^2) in ct12;
sqrt((x-x2)^2+(y-y2)^2)-sqrt((x-x0)^2+(y-y0)^2) in ct20;
end

contractor-list pinter
  for i=1:3;
    dtoaeq(i)
  end
end
contractor propInter
 propag(pinter)
end
contractor isThick
  maxdiamGT(20)
end
```

To our knowledge, it is the first time that such a function is evaluated. This map highlights the emitter positions for which the TDOA passive location error is the most important. These emitter's positions are shown to be located over complex regions really difficult to predict

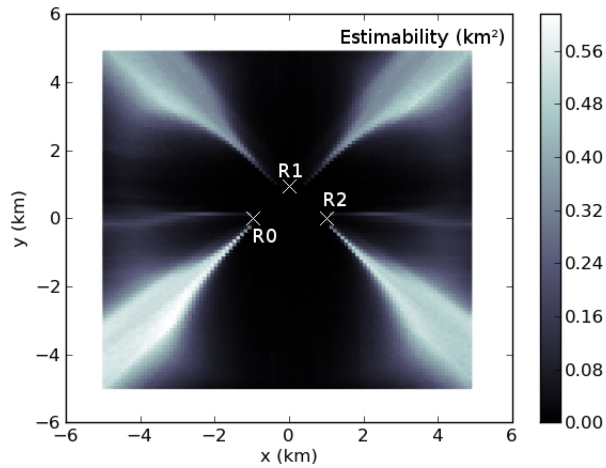



Figure 4. TDOA estimability: receivers are sketched with white crosses: R0 (-1000, 0) m, R1 (0,1000) m and R2 (1000,0) m.

because of the nonlinear hyperbolic equations. These intrinsic properties of $f$ are very useful to properly design passive location systems.

## 5. Conclusion

We have introduced a new interval-based method to evaluate the estimability and shown that it is possible to predict the accuracy of the parameter estimation of a nonlinear model in the case of noisy data. Our method differs from the Cramer-Rao Lower Bound (CRLB) approach, because we have not built a statistics-based estimator. Unlike CRLB, no special assumption is required on the bias or the linearity of the model, neither on the additive noise.

Our approach is not another sensitivity analysis to study the influence of the variation of the parameters on the function's result: $\xi_f$ directly evaluates the error of parameter estimation from $f$ and additive noise set $\mathbb{E}$. Estimability function $\xi_f$ does not require global identifiability. Besides, its use is not restricted to small additive noise. This is due to evaluation method based on interval analysis and set inversion. Application to passive location illustrates the relevance of our approach. We are certain that numerous experimental design problems can be solved thanks to $\xi_f$.

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